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CONTENTS

SLIDES BY IAN ENTWISTLE (C19)

This issue the slide supplement features four slides by Dr. Ian D. Entwistle. To me they have a distinctive style, remarkable when you think that these images are not composed at the whim of the artist, but by Mathematics herself. Ian writes:

At long last my first run of slides came through. I have selected four representative types. They were all run at 640 x 512 and even on the monitor no scan lines are visible. On projection they seem to be OK. I am having them copied on the Canon laser copier directly to A4 or even A3!

2500: Julia set with binary decomposition and minimisation. At 0.32+0.043i, dwell 100.

2501: Rather a complex story to this one, but it is essentially a variation of a cosh $z + \mu$ Julia set. I call it the "Cross of San Marco". Scale 3,3.

2502: "M" at -2.1-1.25i magnification 100 with minimisation.

2503: Newton's method roots for x^5-1 using extended precision. -0.8999-0.9001i ... 1.8+1.8i.

Two recent pieces of hardware for the Acorn Archimedes computer that attract me are FPU and ARM3 chip. The latter raises the performance of almost every aspect by 3-4 times and easily averages 12 MIPS. If code is improved, up to 20-25 MIPS should be possible. This will then compete with even the fastest accelerated Mac IIci.

THE TAMING OF THE SHREW
Part I

A. G. Davis Philip and Kenelm W. Philip

One can always navigate around the Mandelbrot Set by using real and imaginary coordinates, but for many purposes it is also useful to be able to assign unique names to some of the more obvious features. We have been investigating certain details of the Mandelbrot Set for some time now and in the course of this work we have come up with *a rule* (already known by Mandelbrot and others) for determining the number of cycles for each of the appendages to the set, commonly called buds or radicals, and *a naming system* which assigns unique names to each radical.

If one looks at a printout of the Mandelbrot Set at a magnification of one (see Fig. 1, next page), there are several prominent features which have been given colloquial names. Pointing to the left, is the SPIKE on which can be found many MIDGETS. The spike extends from the HEAD which is attached to the BODY. Along the head and the body can be found a number of objects which have been referred to as buds or RADICALS, (the latter term uses a chemical analogy formulated by Mandelbrot). We will use the term radicals to refer to these features (note that in Mandelbrot's terminology the circular appendage to the Set is called an ATOM, while the term Radical refers to the atom *and* the superstructure of

rays and tendrils attached to it). As an aid in naming features one can use the directions of the compass. The spike points to the West; the topmost radical is the North Radical; the bottom radical is the South Radical; the largest radical on the top, left of the North Radical is the NW Radical; the largest radical on the top, right of the North Radical is the NE Radical. There is a series of "heads" that appear to the left of the main head and we denote this series as H, H', H" etc. Other prominent features are the VALLEYS which are found between head' and the head, the head and the body, underneath each radical, and at the East end of the set. The main valley, found between the head and the body, is SEA HORSE VALLEY. The valley at the far east is EAST VALLEY (or ELEPHANT VALLEY, named after the procession of trunk–to–tail elephants which can be seen marching out of this valley). The other valleys are named by the radical that they are under. The entire Set is surrounded by an infinite number of outlying midgets.

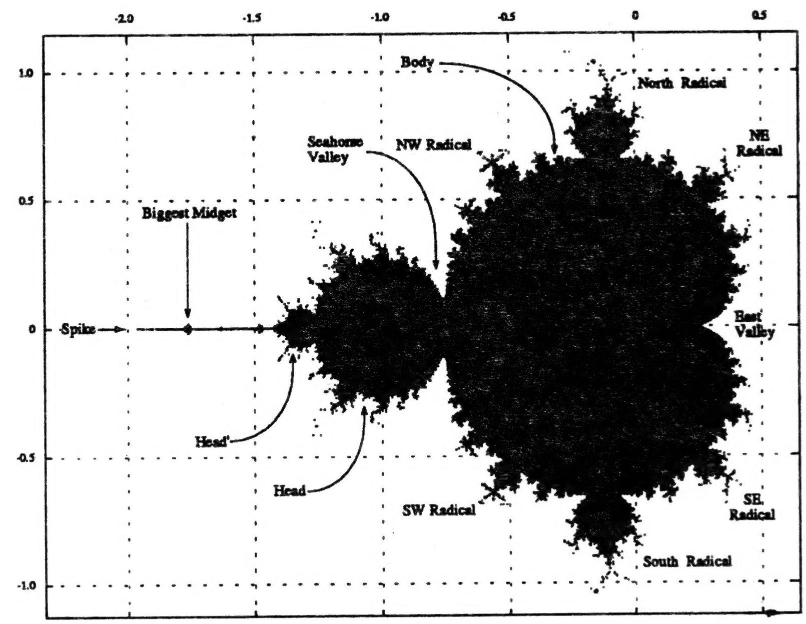

Figure 1 — The Mandelbrot Set

Leading outward from various parts of the Set are linear features that have been called Tendrils, Filaments, or Links. Using the chemical analogy Mandelbrot prefers the name LINKS for connections between outlying features and we will follow his suggestion. Off the tips of atoms can be found star-like patterns and we will call the lines radiating out from the center of such a star RAYS. Some rays continue on out beyond the others in the pattern surrounding the central star and when they do so the part of the ray outside will be called a link.

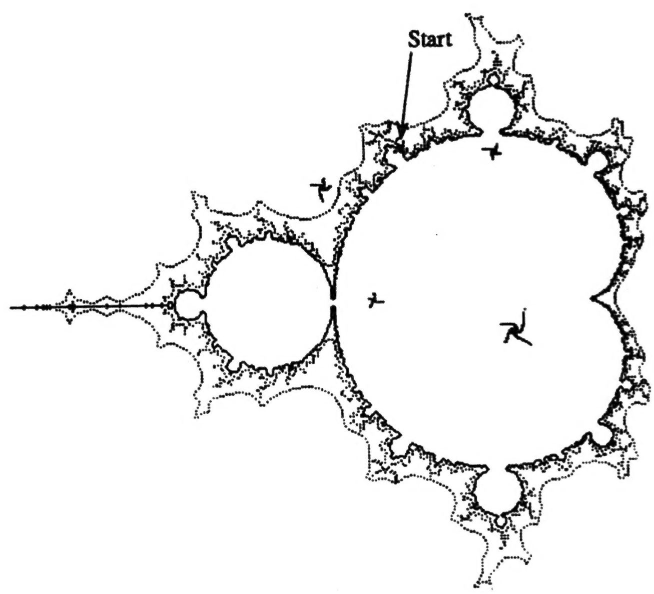

Figure 2a
Iteration track for point in NW Radical
Cycle number = 5

It is well known that all elements of the Set are characterized by a CYCLE number, which for any point in the Set is equal to the number of fixed points in the iteration track. That is: for any value of C in the equation $Z \leftarrow Z^2 + C$, the successive values of Z as the equation is iterated will asymptotically approach one or more stationary points in the complex plane, provided that C lies within the Mandelbrot Set. Fig. 2 shows two such iteration tracks, for a point within the NW Atom (Fig. 2a) with a cycle number of five, and for a point within the body of the Set just under the NW Atom (Fig. 2b) with a cycle number of one. In both figures the starting point is indicated by an arrow. (All points in the body, in fact, have cycle numbers of one, and all points within the NW Atom have cycle numbers of five. Note that

the iteration track in Fig, 2b has five spiral arms, which is equal to the cycle number of the atom just above the starting point—this interesting relationship will be explored in Part II of this paper.)

One does not have to examine iteration tracks to ascertain the cycle number of an atom, since the number of rays in the star attached to the tip of the atom (counting the link from the star to the atom as one ray) turns out to be equal to the cycle number for the atom in question. This convenient equality allows one to determine the cycle number of any radical whose rays can easily be counted by simply looking at the image of the radical and its star.

KWP has found a delightfully simple relationship which lets you predict in advance the ray # and series interval (= difference in ray #s of two adjacent radicals in the same 'series'—see below) of any radical, as follows:

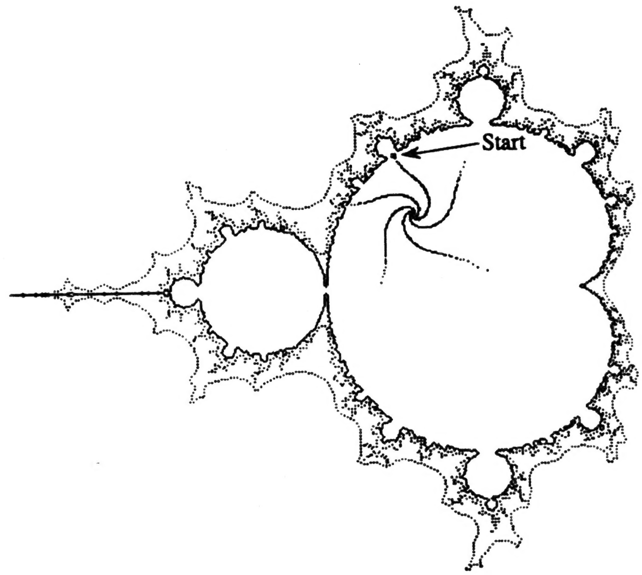

Figure 2b
Iteration track for point under NW Radical
Cycle number = 1

The Primary Series (Head, N Radical, NE Radical, etc. down into E Valley) has ray #s increasing by one. The head has two rays (one pointing outward and one pointing in towards the head), the N Radical has three rays, etc. If you look at the largest radical between any two radicals in the primary series, the ray # is given by the sum of the ray #s of the two primary radicals on either side. For example, the NW Radical has a ray # of 5 (R# = 3 + 2), and is the first radical in the Secondary Series which runs towards the head with ray #s increasing by two. The NNW Radical has a ray # of 8, given by 5 + 3 (5 for the NW Radical, 3 for the N Radical). Furthermore, the radicals to one side of this largest intermediate radical form a series where the ray # increases by a number equal to the ray # of the radical that the series is heading towards. For instance, the radicals running from the NNW Radical towards the N Radical form a series with the ray # increasing by 3; while the radicals running from that same NNW Radical towards the NW Radical form a series with the ray # increasing by 5. This system has been checked 6 levels down by actual ray # counts. It should be noted that Mandelbrot (of course!) discovered the ray # (= cycle #) relationship long before KWP stumbled on it a few months ago, as he (Mandelbrot) told AGDP during a visit to Union College (and the same relationship is alluded to in the caption and text references to Fig. 34 in *The Beauty of Fractals*, page 61, where it states that index(c) establishes a Fibonacci partition in M). However, his formulation is more complex than the one given here. It involves assigning a fraction to each radical; adding the fractions gives the value of the next radical inbetween two majors.

One fascinating question to which the answer was not known by us until recently is the following: You could simply name radicals by their ray # and the series interval (call it 'delta'). Then the N Radical would be 3(1), the NE Radical would be 4(1), and so on. The NW Radical would be 5(2), the next major radical to the west would be 7(2), and so on down into Seahorse Valley. The NNW Radical would be 8 (3), and the largest radical to its east would be 11(3), and so on heading under the west base of the N Radical. This system with the ray # followed by the delta in ( ) actually tells you something about the radical, though not where to find it! The question is: If this system is used, are there any radicals with the same name? KWP recently checked over 20,000 radicals lying between the first 20 Primary Radicals with a MS BASIC program that used the ray numbers and deltas as x/y coordinates and plotted the values checking for coincidences with the POINT statement (using the rule above to generate the values) and found that all the ray number/delta pairs produced were unique. This preliminary result suggested, but did not prove, that all ray #/delta names might indeed be unique. More recently, however, R. W. Gatterdam of the

University of Alaska (Fairbanks) Mathematics Department was kind enough to work up a proof that ray#/delta names generated by the above system are in fact unique. On the other hand, even though ray #/delta names are unique, they do not help one *find* the named radical. For that purpose we turn to an alternative scheme based on the route one travels to reach a given radical.

One of us (AGDP) has formulated a system which ties in very nicely with the rule expressed in the paragraphs above. (We have needed such a system to keep track of all the images we are building up on our various storage media!) Refer to Fig. 3 for a location guide to Figs. 4-7, and turn to Fig. 4, which shows the upper half of the Mandelbrot Set, and we can now start a naming process for the radicals. First let us consider the idea of the DOMAIN of a Radical. The domain of Radical 2 is the surface of the Mandelbrot Set between Radical 2 and Radical 3. (One can also consider the body of the Set to be R1, with a domain running from the Real axis at +0.25 to the Real axis at -0.75.) The domain of Radical 3 is the surface between Radical 3 and Radical 4, and so on. All the minor radicals fall in the Domain of some larger radical and it aids in naming the radical in question to know which domain it falls in. In naming the primary radicals, one can imagine going along an arc, above the set, from Radical 2 to 3 to 4 and so on into East Valley. The head is Radical 2. The North Radical is Radical 3. The NE Radical is Radical 4. One can continue into East Valley, naming the next largest radical to the right in order, R5, R6, R7 etc.

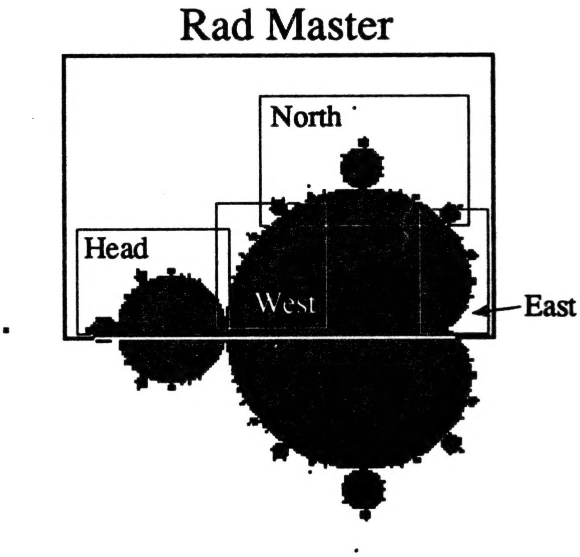

Figure 3
Index to Rad Masters

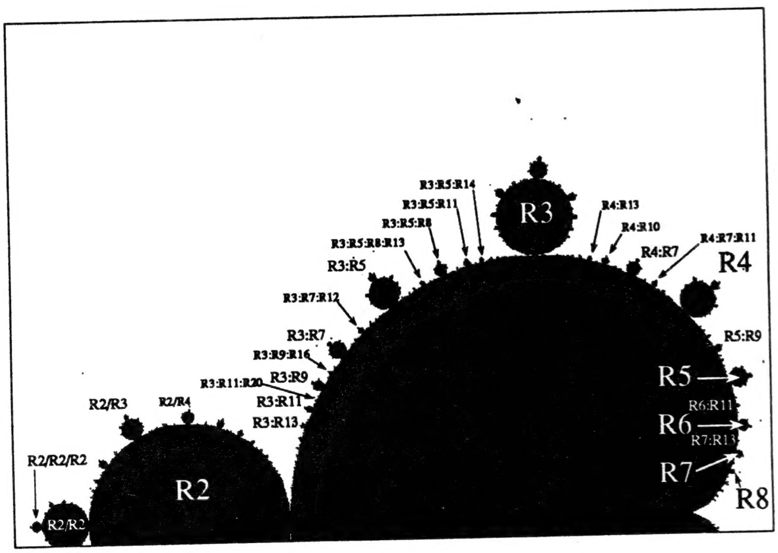

Figure 4
Rad Master

Note that there is another set of radicals, extending from Radical 3 towards Radical 2. The largest one in between R2 and R3 is the NW Radical. You can imagine following the arc from R2 to R3, reversing and then going to the NW Radical. To determine the radical number, we use the Rule (R2 + R3) which would give us 5, or R5. The full name of this radical contains all the major turning points as we follow the arc. Since *all* the arcs start at R2 we may drop the R2 at the beginning of each sequence while we are considering radicals on the body, although strictly speaking every radical name begins with 'R2:' or 'R2/'. The arc starts at R2, goes to R3 and reverses to R5 (NW Radical)—its name would be R3:R5. The next major radical, going left (toward the head) has R2 + R5 which would give R7 and its name would be R3:R7. Since you continue along the arc in the same direction from the N Radical past R3:R5 you can just

substitute R7 for the R5. This pattern continues all the way into the Sea Horse Valley; the last R number goes up by 2 for each Radical.

If we move back to R3:R5 we can note that there is another series of radicals to be found going off to the right into the valley under the Radical R3. The largest Radical between R3:R5 and R3 has 5 + 3 rays, and since the direction along the arc has to be reversed again to reach it, the name would be R3:R5:R8. The next largest radical has 8 + 3 rays and would be R3:R5:R8:R11. Each radical in this series increases by 3 and you would find the fifth R number going to R14, R17, R20 etc.

The routine outlined in the above paragraphs can be followed to any depth that you wish and each radical will end up with a unique name. There will be cases where the last R number in the series is the same as a radical number in some other series, but the R numbers earlier in each series will be different. The delta can always be appended in parentheses to give a *complete* designation of a given radical. Note that as long as you continue on the same surface a colon is used between the R numbers. If we wish to name the radicals which are found on atoms we use the "/" sign to indicate this change of surface.

If one investigates the ray structure leading from radicals on atoms it will be found that the situation is more complicated, but still subject to being named under this system. First, one should note that a higher order radical has a *series* of ray structures, which in effect provide a history of the route taken to reach that radical. The outermost ray structure has a ray number characteristic of the first-level radical: thus all radicals on the head (R2) end with a ray number of two, and all radicals on the North Atom (R3) end with a ray number of three, and so on.

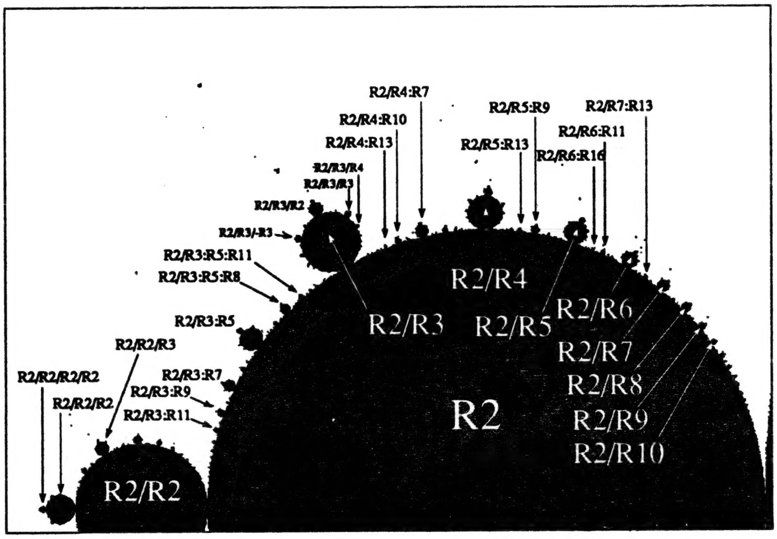

Figure 5 — Rad Master Head

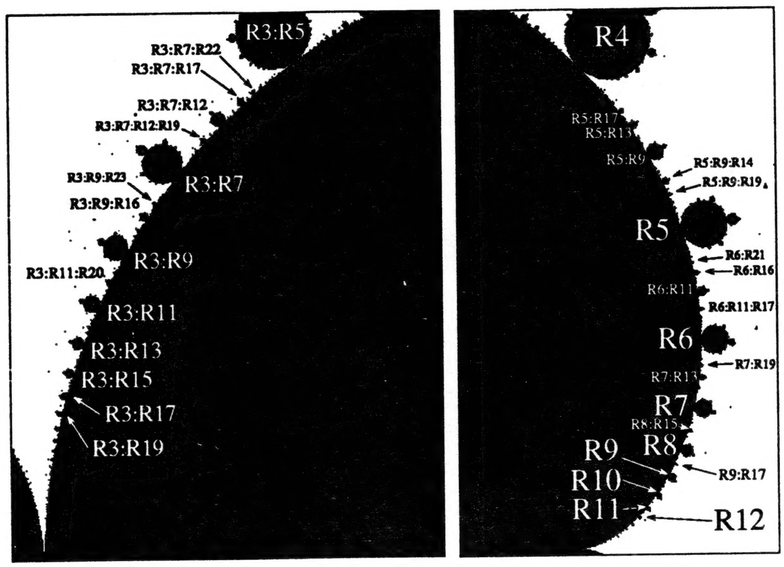

Figure 6 — Rad Master East/West

Beneath that outer ray structure is another point from which rays diverge, and this point obeys the standard rule, now controlled by the position of the higher-order radical on the atom to which it is attached. For example, the northeast radical on the Head has a compound ray structure with a lower ray number of three, and an outer ray number of two: its name would be R2/R3. The northeast radical on *that* atom has a triple ray structure: lower and center with a ray number of three, outer with a ray number of two: its name would be R2/R3/R3. (For those who may try following this article with their favorite Mandelbrot program, note that increased magnification on a ray center never reveals a central midget—which fact will serve to distinguish ray centers from other features in the radical.)

Figure 8 shows the ray structure for R2:R6/R5/R4/R3, at

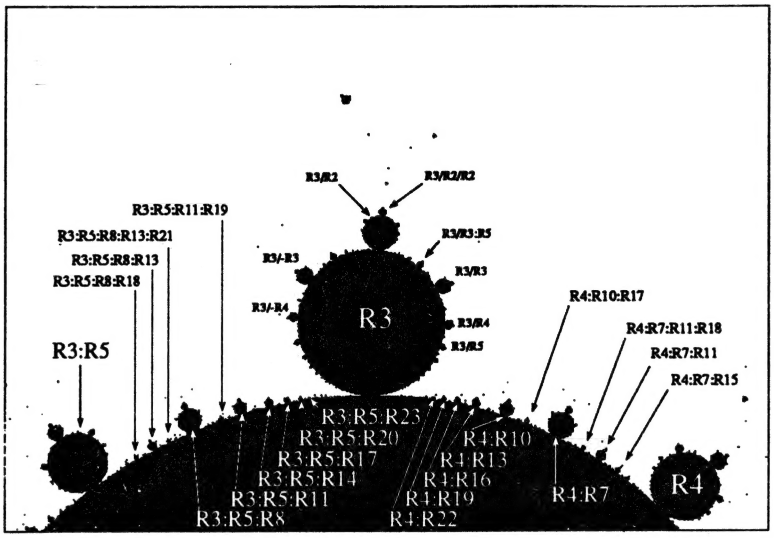

Figure 7 — Rad Master North

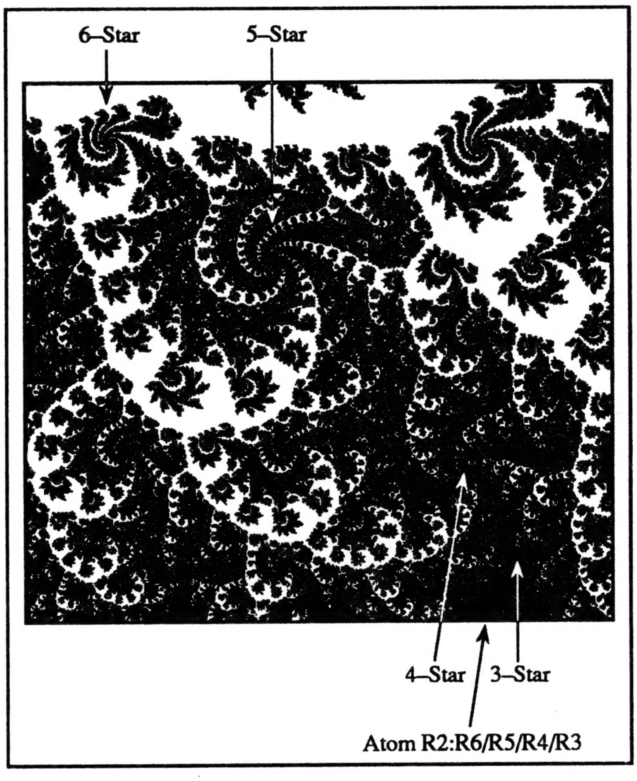

Figure 8
Ray Structures above Atom R2:R6/R5/R4/R3

the east end of the Set. One can see stars with each of those ray numbers in the radical.

One should note that, when 'compass direction' names are used, we assume each radical, atom, or midget is in the "rectified position", i.e. it is pointing to the west. Then, when talking about features on it, we can use the common names of N Radical, NW Radical and all will know which feature is meant, independent of the original orientation.

The above system can be used to give a unique name to every radical that appears in the Mandelbrot Set. When dealing with radicals on the bottom of the set we use a minus sign. Thus the lower large radical, South Radical, would be labeled as -R3, for example. For radicals on atoms, we count them positive going clockwise from the end of the radical, and negative going counterclockwise. The largest subradical on the North Atom, west of the north end, is thus:

R2:R3/-R3

Figures 3-7 present this naming system for many of the radicals visible on a 1x image of the upper half of the entire Mandelbrot Set. Figure 3 is a location diagram for the four following figures, which display names for the entire upper half of the Set (Fig. 4), and for four somewhat magnified regions from that image: the Head (Fig. 5), the West and East portions of the body (Fig. 6), and the North portion of the body (Fig. 7). Radicals on atoms are shown, in addition to radicals on the body of the Set. These figures, in conjunction with the text above, should make it easy for anyone to produce names on this system for any radical on the Mandelbrot Set.

Acknowledgements

We would like to thank Homer Smith for an enlightening discussion with AGDP, which clarified the situation with regard to radicals on radicals; and Scott Huddleston for reading a draft of this paper, and suggesting ideas which we intend to develop in Part II.

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TUTORIAL — COMPLEX ANALYTIC GEOMETRY

Is complex analytic geometry really complex? According to an aphorism of B. B. Mandelbrot, "to simplify the problem, complexify the variables!" This is certainly true of the roots of a polynomial, or of the study of smooth functions, or iteration. It is also true of analytic geometry.

The equation of a curve or *locus* is usually expressed as a relation between x and y. It can also be expressed as a relation between z and $\bar{z}$, sometimes to distinct advantage. The important thing is that one complex equation is ordinarily equivalent to two real equations; to obtain a genuine locus these equations should be essentially the same.

For example, in complex terms the equation of a circle is $|z-a|=r$, which can be rewritten $(z-a)\overline{(z-a)}=r^2$. This equation is invariant under complex conjugation, which indicates that it represents a single real equation.

A straight line in the complex plane can be represented by the parametric equation $z=a+bt$, where a and b are complex numbers and $b \neq 0$; the parameter t runs through all real values. Two equations $z=a+bt$ and $z=a'+b't$ represent parallel lines whenever b' is a positive multiple of b. They represent the same line whenever the quantities $a'-a$ and b' are real multiples of b, and they proceed in the same direction if b' is a positive multiple of b. The direction of $z=a+bt$ can be identified with arg b. The angle between $z=a+bt$ and $z=a'+b't$ is $\arg\frac{b'}{b}$. Note that it depends on the order in which the lines are named. The lines are orthogonal (at right angles) if $\frac{b'}{b}$ is pure imaginary.

Problems such as finding the intersection between lines and circles, parallel or orthogonal lines, tangents, and so forth, tend to become simple when expressed in complex form.

The inequality $z-a<r$ describes the inside of the circle of radius r. Similarly, a directed line $z=a+bt$ determines a right half-plane consisting of all points z with $\Im\frac{z-a}{b}<0$ and a left half-plane with $\Im\frac{z-a}{b}>0$. It is easy to show that this distinction is independent of the particular parametric representation.

EXERCISES

1. When does $az+b\bar{z}+c=0$ represent a straight line?
2. Write the equation of an ellipse, a hyperbola, a parabola in complex form.
3. Prove that the diagonals of a parallelogram bisect each other, and that the diagonals of a rhombus are orthogonal.
4. Prove analytically that the midpoints of parallel chords of a circle lie on a diameter perpendicular to the chords.
5. Show that all circles that pass through a and $1/\bar{a}$ intersect the circle $|z|=1$ at right angles.

FRACTALS IN CARDIOLOGY

excerpts from a letter

—*Kenneth M. Stein, M.D.*

Ary Goldberger has been a leader in applying the techniques of nonlinear dynamics to cardiology. Fractal geometry has been used to study both cardiac morphology and cardiac dynamics. In the first instance, Goldberger has shown that fractal models of the electrical system of the heart can generate realistic simulations of the normal human electrocardiogram.

In the case of cardiac dynamics there has been a lot of interest in studying human heart rate variability. It is a longstanding clinical observation that people with "healthy" hearts have more variability in their pulse rates than do those with "unhealthy" hearts (I am being deliberately vague here). Since the intervals between successive heart beats vary in an irregularly irregular manner a number of researchers (myself included) have been prompted to look for a fractal structure that describes the behavior. However, to this point there is no clear data that would support or refute this claim.

Another application of fractal geometry to cardiology has been in the study of abnormal ("ectopic") heartbeats. My lab has shown that the distribution of a certain type of abnormal beats (known as "premature ventricular contractions" or "PVCs") over time can be characterized as a fractal dust of dimension less than one. Furthermore, in a study of 20 patients, the value of this dimension was inversely correlated with survival, suggesting that this technique can be used as a prognostic tool. Two papers regarding these findings are in press and a third is under review.

TWO LETTERS FROM A NINTH GRADER

—Feb 14, 1989

I am a ninth grader in high school, and I am preparing an oral presentation on fractals for my Advanced Math/Calculus class. I read an article, "Computer Recreations", in SCIENTIFIC AMERICAN (Feb. 1989). It mentioned that if I were to send a self-addressed stamped envelope, I could receive a full set of transparencies. I have enclosed a self-addressed stamped envelope for the transparencies. I would also like to subscribe to your newsletter, AMYGDALA. If you have any additional information that you would send to me, I would greatly appreciate it. Thank you for your consideration in helping me with my presentation.

* * * * *

I am writing to thank you for your help in my presentation

on fractals. I'm sorry I took so long to write back, but I thought that you might be interested in hearing how I did. I presented for two days (or a total of about eighty minutes) and my teacher was extremely impressed. I covered many topics, including the Mandelbrot Set, fractal dimensions, fractals in nature, IFS codes, the Collage Theorem, and the Random Iteration Algorithm. I received the highest grade in the class, and my teacher mentioned that my project might be the best he's ever had in eighteen years of teaching. I showed the slides to the class, and they were extremely impressed with them. I have enclosed a ten dollar bill as payment for the slides. Anyway, thanks once again for your extremely useful help in my presentation.

— Received, May 10, 1989

BIBLIOGRAPHY

129. CA Pickover, **"A Note on Rendering Chaotic 'Repeller Distance Towers'"**. *Computers in Physics* [May/June 1988] 75-76. [Lovely colour photos of towers for $\Omega \rightarrow \cosh(\Omega) + \mu$ and $\Omega \rightarrow \Omega^2 + \mu$. No algorithms so for me [Entwistle] it is the lovely effects that appeal.]

130. **"Chaos and Order in Nature"**. Volume 11 in the Springer Series in Synergetics (See #129).

131. R Shaw, **"Modeling Chaotic Systems"**. *Chaos and Order in Nature* (See #130) 218-231. [Introducing the concept of deterministic chaos, with a focal point on information theory.]

132. RHG Helleman, **"Feigenbaum Sequences in Conservative and Dissipative Systems"**. *Chaos and Order in Nature* (See #130) 232-248. [Period doubling route to chaos and the concept of renormalization.]

133. M Diener & T Poston, **"On the Perfect Delay Convention or the Revolt of the Slaved Variables"**. *Chaos and Order in Nature* (See #130) 249-268. [Catastrophe theory: the subject is well illustrated by graphs.]

134. **"Evolution of Order and Chaos in Physics, Chemistry and Biology"**. Volume 17 in the Springer Series in Synergetics (See #129).

135. T Geisel & J Nierwetberg, **"Scaling Properties of Discrete Dynamical Systems"**. *Evolution of Order and Chaos in Physics, Chemistry and Biology* (See #134) 187-196. [Universal constants in period doubling bifurcation.]

136. CA Pickover, **"A Note on Rendering 3-D Strange-Attractors"**. *Comput. & Graphics* 12,2(1988) 263-267. ["The purpose of this note is to illustrate a very simple graphics technique whereby chaotic attractors can be clearly visualized."]

137. BB Mandelbrot, **"Self-Inverse Fractals Osculated by Sigma-Discs and the Limit Sets of Inversion Groups"**. *The Mathematical Intelligencer* 5,2(1983) 9-17.

ON "AMYGDALA"

—*Reinald Eis*

You may be interested to learn that the word "amygdala" was first used in the ancient *Greek* (not Latin) language. The Romans later adopted it as a foreign word for their "nux graeca", which is what almonds used to be called in Latin originally.

"αμνγδαλη" stood the test of time and it is still in use in modern Greek.

RENEWAL

For 10 of you subscribers out there this is the last issue of your current subscription. For 469 of you it is the next to last issue. I urge you to use the enclosed form to renew your subscription promptly to avoid missing anything.

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